# ABELIAN TORSION GROUPS HAVING A MINIMAL SYSTEM OF GENERATORS(1)

BY

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NOTATION AND TERMINOLOGY. Let $T$ denote an arbitrary Abelian torsion group. Let $G$ denote an arbitrary primary $p$-group to be considered fixed in each separate proposition. Let the symbol "iff" mean "if and only if," $<$ mean properly contained in, $\subset$ mean contained in, $N\backslash M$ mean the set of elements in $N$ and not in $M$, $\cong$ mean isomorphic to, $(N_a)_{a\in A}$ denote a family of sets, elements or groups—as the case may be—$N_a$ indexed by members of an index set $A$; and if for each $a\in A$ $N_a$ is a group let $\oplus_{a\in A} N_a$ denote the direct sum of the $N_a$'s—$\oplus$ denotes a direct sum. If $x\in G$ let $h(x)$ denote the ordinary height of $x$—see [2].

If $S$ is a subset or subgroup of $T$, let $|S|$ denote the power—cardinal number—of $S$. Let $(S)$ mean the same thing as $S$, and $\{S\}$ mean the subgroup generated by the elements of $S$. If $x_1, x_2, x_3, \cdots \in T$, let $(x_1, x_2, x_3, \cdots)$ denote the set whose only elements are $x_1, x_2, x_3, \cdots$, and let $\{x_1, x_2, x_3, \cdots\} = \{(x_1, x_2, x_3, \cdots)\}$.

DEFINITION. A subset $S$ of elements of $T$ is said to be *a minimal system of generators* of $T$—abbreviated by (m.s.g.)—iff $S$ generates $T$, and no proper subset of $S$ generates $T$.

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## I. On primary groups.

REMARK. Since most of the following propositions are trivially true for finite groups, we shall often assume while proving them that $G$ is infinite.

LEMMA 1. *Let $G$ have a (m.s.g.) $S$, and let $x\in S$. Then $h(x)=0$.*

**Proof.** Assume for some $x\in S$ $\exists y\in G$ such that $py=x$. Then $\exists$ $x_1, x_2, \cdots, x_m\in S$ and integers $n_1, \cdots, n_m$ such that $y=n_1x_1+\cdots+n_mx_m$. By the minimality of $S$ we may assume that $x_1=x$ and $n_1\neq 0$. Then $x=py=n_1px+n_2px_2+\cdots+n_mpx_m$. Thus $(S\backslash(x))\cup(px)$ is a (m.s.g.) of $G$. Continuing this with $p^{i+1}x$ in place of $p^ix$ we arrive finally at $(S\backslash(x))\cup(p^nx)$ is a (m.s.g.) of $G$ where $p^nx=0$, which is impossible.

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LEMMA 2. *Let $G$ have a (m.s.g.) $S$. Suppose that $x=n_1x_1+\cdots+n_mx_m$, where $x_1,\cdots,x_m\in S$ and for some integer $i$, $1\leqq i\leqq m$, $(n_i, p)=1$. Then $(S\backslash(x_i))\cup(x)$ is a (m.s.g.) of $G$.*

**Proof.** We may take $i=1$. That $(S\backslash(x_1))\cup(x)$ generates $G$ is trivial. To show it is minimal it suffices to show that no $x_i$ for $1<i\leqq m$ is a linear combination of elements of $(S\backslash(x_1, x_i))\cup(x)$. So assume one such $x_i$, say $x_m$, is a linear combination of the elements of $(S\backslash(x_1, x_m))\cup(x)$. Then $\exists$ elements $y_m,\cdots,y_k$ of $S$ different from the $x_i$'s and integers $l_1,\cdots,l_k$, such that $x_m = l_1x + l_2x_2 + \cdots + l_{m-1}x_{m-1} + l_my_m + \cdots + l_ky_k$. Thus $x_m = l_1n_1x_1 + (l_1n_2+l_2)x_2+(l_1n_3+n_3)x_3+\cdots+(l_1n_{m-1}+l_{m-1})x_{m-1}+l_1n_mx_m+l_my_m+\cdots+l_ky_k$. Since $S$ is minimal, the coefficients of all the $x$'s and $y$'s on the right hand side of the last equation which are not equal to $x_m$ must be multiples of $p$. In particular, $l_1n_1$ is a multiple of $p$. But $n_1$ was assumed to be relatively prime to $p$. Hence $l_1$ is a multiple of $p$. Hence also the coefficient $l_1n_m$ of $x_m$ is a multiple of $p$. Thus $h(x_m)>0$, contrary to Lemma 1.

LEMMA 3. *Let $G$ have a (m.s.g.) $S$. Then $x\in G$ can be extended to a (m.s.g.) of $G$ iff $h(x)=0$.*([2])

REMARK. Before stating our next theorem, we remark that all basic subgroups of a primary group are isomorphic; and for nonprimary groups while I of the following theorem is implied by VI, I does not in general imply VI, as is the case in a group which is the direct sum of an infinite number of cyclic groups having different primes for orders.

THEOREM 1. *Let $G$ be a primary $p$-group. Then the following statements are equivalent:*

(I) *$G$ has a minimal system of generators.*

(II) *Some system of representatives of some basis of $G/pG$ generates $G$.*

(III) *$G$ is finite or $G/pG$ has the same power as $G$.*

(IV) *$G$ has the same power as a basic subgroup of itself.*

(V) *$G$ has a direct summand which is a direct sum of cyclic groups and has the same power as $G$.*

(VI) *$G$ has a minimum system of generators.*

(VII) *$G$ is finite or the automorphism group of $G/D$ has power equal to that of the set of all subsets of $G$, where $D$ is the divisible part of $G$.*

(VIII) *$G$ is finite or the automorphism group of $G/D$ has power exceeding that of $G$, $D$ being as in* (VII).

**Proof of** (I) **implies** (II). Assume $G$ has a (m.s.g.) $S$. Then if $\sigma$ is the natural homomorphism of $G$ onto $G/pG$, certainly the set $(\sigma(x))_{x\in S}$ generates $G/pG$, and it remains to show that this set is independent. So assume that for some $x\in S$, $x+pG=(n_1x_1+pG)+\cdots+(n_mx_m+pG)$, where each $x_i\in S$, and

[2] *Added in proof.* Lemma 3 has an analogue for an arbitrary Abelian group $G$, in which case Lemma 1 is best proved by observing that $G/\{S\backslash(x)\}$ is cyclic.

$x_i \neq x$. Then $\exists g_1, \cdots, g_m \in G$ such that $x = n_1x_1 + \cdots + n_mx_m + p(g_1 + \cdots + g_m)$. Then by Lemma 2 the set $(S\backslash(x)) \cup (p(g_1 + \cdots + g_m))$ is a (m.s.g.) of $G$. This contradicts Lemma 1 since $h(p(g_1 + \cdots + g_m)) \geqq 1$.

**Next we prove** (II) **implies** (III) **implies** (IV). If $G$ is finite there is nothing to prove. So assume $G$ is infinite; then since some system of representatives of some basis of $G/pG$ generates $G$, we must have $|G| = |G/pG|$. But by a theorem of Kulikov which holds for an arbitrary primary group $G$ and any basic subgroup $B$ of it we have $G/pG \cong B/pB$. Thus $|G| = |G/pG| = |B/pB|$. Since $|G| \geqq |B| \geqq |B/pB|$, we obtain $|G| = |B|$. This completes the proof of (II) implies (III) implies (IV).

**Proof of** (IV) **implies** (V). Write $G = L \oplus D$ where $D$ is divisible and $L$ is reduced, and let $B$ be basic in $L$. Then $B$ is basic in $G$, so that $|G| = |B| \leqq |L| \leqq |G|$. Thus we may assume that $G$ itself is reduced, since this would be the same as showing $L$ has a direct summand as desired.

We may assume that $G$ is infinite.

*Case* 1. $G$ is countable.

Any countable reduced $p$-group is known to have a direct summand which is a direct sum of a countable number of cyclic groups. See [1, p. 143].

Now let $B = \oplus_{i=1}^{\infty} B_i$, where $B_i$ is a direct sum of cyclic groups of order $p^i$, and $B$ is a basic subgroup of $G$.

*Case* 2. $|G| > \aleph_0$, and $\exists$ an integer $n$ such that $|\oplus_{i=1}^{n} B_i| = |G|$.

It is then known that—see [1, p. 98, Baer's Theorem]—$G = (\oplus_{i=1}^{n} B_i) \oplus \{\oplus_{i=n+1}^{\infty} B_i, p^nG\}$, and in this case we may take our direct summand to be $\oplus_{i=1}^{n} B_i$.

*Case* 3. $|G| > \aleph_0$, $G$ has no elements of infinite height, and for no integer $n$ is $|\oplus_{i=1}^{n} B_i| = |G|$.

Then there is no greatest $|B_i|$, and in this case $|B| = \sum_{i=1}^{\infty} |B_i|$, and $|B|$ is a "limit" cardinal. Let $i_1$ be the first integer such that $|B_{i_1}|$ is infinite, and assume $i_n$ has been defined. Let $i_{n+1}$ be the first integer larger than $i_n$ such that $|B_{i_{n+1}}| > |B_{i_n}|$. Then $\sum_{n=1}^{\infty} |B_{i_n}| = |B| = |G|$.

Now if we denote by $\overline{B}$ the closure of $B$, i.e., the torsion subgroup of the strong direct sum of the $B_i$'s, then—since $G$ has no elements of infinite height—it is well known that $G$ may be considered as a pure subgroup of $\overline{B}$—see [1, p. 112]. Then $G/B$ is divisible and of power not exceeding that of $B$. Let $S$ be a system of representatives of the nonzero cosets of $G/B$. Then each element of $S$ is a sequence $(v_1, v_2, v_3, \cdots)$ with $v_i \in B_i$, and where an infinite number of the $v_i$'s are different from zero. Moreover, any element differing from $(v_1, v_2, v_3, \cdots)$ in only a finite number of places belongs to $G$ and represents the same coset of $G/B$ as $(v_1, v_2, v_3, \cdots)$.

Now $S$ may be divided into a sequence of disjoint subsets $S_1, S_2, S_3, \cdots$ such that $|S_j| \leqq |B_{i_j}|$, for $j = 1, 2, 3, \cdots$.

By our last remark, we may assume in addition that each element of $S_j$ has its first $i_j$ components equal to zero. Now since each $|B_{i_j}|$ is infinite, every $B_{i_j}$ is a direct sum of cyclic groups $\oplus_{a\in A_{i_j}}\{v_a\}$ for some index set $A_{i_j}$ where $|A_{i_j}|=|B_{i_j}|$.

Now if $x\in S$ has some nonzero component from $B_{i_j}$, then $x\in S_l$ where $l<j$; moreover this component is of the form $k_1v_{a_1}+k_2v_{a_2}+\cdots+k_mv_{a_m}$ for some integers $k_1,\cdots,k_m$ and $a_t\in A_{i_j}$, for $t=1,\cdots,m$. Thus the set $C_{i_j}$ of all $a\in A_{i_j}$ such that $v_a$ occurs in an expression of a component of some element of $S$ is of cardinal $\leqq\aleph_0(\sum_{l<j}|B_{i_l}|)=\aleph_0|B_{i_{j-1}}|=|B_{i_{j-1}}|<|B_{i_j}|$.

Now $B_{i_j}=(\oplus_{a\in C_{i_j}}\{v_a\})\oplus(\oplus_{a\in(A_{i_j}\setminus C_{i_j})}\{v_a\})$ where $|A_{i_j}\setminus C_{i_j}|=|B_{i_j}|$. Also $G$ is generated by $S$ and $\oplus_{i=1}^{\infty}B_i$. Moreover it is clear from the above that $G=\oplus_{j=1}^{\infty}(\oplus_{a\in(A_{i_j}\setminus C_{i_j})}\{v_a\})\oplus\{S,\ \oplus_{j=1}^{\infty}(\oplus_{a\in C_{i_j}}\{v_a\}),$ all $B_i$ with $i\neq i_j$ for any $j\}$.

Thus $\oplus_{j=1}^{\infty}(\oplus_{a\in(A_{i_j}\setminus C_{i_j})}\{v_a\})$ is a direct summand as required.

*Case* 4. $|G|>\aleph_0$, $G$ has elements of infinite height, and for no integer $n$ is $|\oplus_{i=1}^{n}B_i|=|G|$.

Let $\overline{G}$ be the subgroup of $G$ consisting of all the elements of $G$ of infinite height. Then by a theorem of Kulikov, [1, p. 103], the image $\overline{B}$ of $B$ under the natural homomorphism of $G$ onto $G/\overline{G}$ is a basic subgroup of $G/\overline{G}$ and $\overline{B}\cong B$. Now $\overline{B}$ and $G/\overline{G}$ satisfy the conditions of case 3—see [1, p. 112]. Hence $G/\overline{G}$ may be written as $\overline{H}\oplus\overline{K}$ where $\overline{H}$ is a direct sum of cyclic groups, $|\overline{H}|\geqq|G/\overline{G}|$, and as in the proof of case 3, $\overline{H}$ is contained in $\overline{B}$. Then let $H$ be the subgroup of $B$ generated by representatives from $B$ of the elements of $\overline{H}$. Then we have $(H+\overline{G})/\overline{G}$ is a direct summand of $G/\overline{G}$, $H\cap\overline{G}=0$, and hence—see [2, p. 18]—$H$ is a direct summand of $G$. Then we have $|H|=|\overline{H}|\geqq|G/\overline{G}|\geqq|\overline{B}|=|B|=|G|$. Hence $|H|=|G|$. Moreover, $H$ being a subgroup of $B$ which is a direct sum of cyclic groups, $H$ is also a direct sum of cyclic groups. This completes the proof of (IV) implies (V).

**Proof of** (V) **implies** (I). Let $G=H\oplus K$, where $|H|\geqq|K|$ and $H$ is a direct sum of cyclic groups. Then we must show that $G$ has a (m.s.g.). So let $B=\oplus_{a\in A}\{v_a\}$ be a basic subgroup of $K$. Then it is readily checked that $pB$ is a basic subgroup of $pK$ and $|pK/pB|\leqq|K|\leqq|H|$. Let $H=\oplus_{b\in Q}\{w_b\}$, and $pK/pB=\oplus_{z\in Z}D_z$, where each $D_z$ is an indecomposable divisible group. Then $Q$ may be divided into $|Z|+1$ sets $B_z$, $B_1$ such that for each $z\in Z$, $|B_z|=\aleph_0$. Let $D_z$ be generated by $d_1^z, d_2^z, d_3^z,\cdots$ where $pd_1^z=0$, and $pd_{i+1}^z=d_i^z$ Let $e_i^z\in pK$ represent $d_i^z$. Now if each set $(w_b)_{b\in B_z}$ is arranged in a sequence $w_1^z$, $w_2^z, w_3^z,\cdots$ where the order of $w_i^z$ is $p^{n_i^z}$, then for each $z\in Z$ form the set

$$E_z=(w_1^z+e^z_{n_1^z},\ w_2^z+e_{n_1^z+n_2^z},\cdots,w_m^z+e^z_{n_1^z+\cdots+n_m^z},\cdots).$$

Then we assert that the set

$$T=\Big(\bigcup_{a\in A}(v_a)\Big)\cup\Big(\bigcup_{b\in B_1}(w_b)\Big)\cup\Big(\bigcup_{z\in Z}E_z\Big)$$

is a (m.s.g.) of $G$. First we observe that the images of the elements of

$$\left(\bigcup_{b\in B_1}(w_b)\right)\cup\left(\bigcup_{z\in Z}E_z\right)$$

under the natural homomorphism of $H+pK$ onto $(H+pK)/pB$ generate $(H+pK)/pB$, and that $\bigcup_{a\in A}(v_a)$ generates $B$ which contains $pB$. Thus $\{T\}\supset\{H+pK, B\}$. Now $\{pK, B\}=K$—see [1, p. 109]. Thus $\{T\}=G$. To show $T$ is minimal, we first observe that $\{T\backslash(w_b)\}$ does not contain $w_b$ for any $b\in B_1$, and that $\{T\backslash(w_m^z+e^z_{n_1^z+\cdots+n_m^z})\}$ does not contain $w_m^z$ for any $z\in Z$ and any integer $m$. Moreover we will show that for no $a_0\in A$ does $\{T\backslash(v_{a_0})\}$ contain $v_{a_0}$.

The last assertion follows from: $\{T\backslash(v_{a_0})\}\subset\{H, pK, (v_a)_{a_0\neq a\in A}\}$ which does not contain $v_{a_0}$, since the images of the $v_a$'s are independent in $G/(H+pK)$, as mentioned above. This completes the proof of (V) implies (I).

**Proof of** (I) **is equivalent to** (VI). We will prove the stronger statement, namely, $S$ is a (m.s.g.) of $G$ iff $S$ is a (M.s.g.) of $G$. It follows at once from the definitions that if $S$ is a (M.s.g.) of $G$, then $S$ is a (m.s.g.) of $G$.

Next assume that $S$ is a (m.s.g.) of $G$, and that for some $n\geqq 1$ the $n$ elements $x_1, \cdots, x_n$ of $S$ may be replaced by the $n-1$ elements $y_1, \cdots, y_{n-1}$ of $G$ in such a way that the resulting set $Q$ still generates $G$. Hence for each $x_i \exists x_1^i, \cdots, x_{m_i}^i\in S\backslash(x_1, \cdots, x_n)$ such that $x_i\in\{x_1^i, \cdots, x_{m_i}^i, y_1, \cdots, y_{n-1}\}$.

Consider the subset

$$T=(x_1, \cdots, x_n)\cup\left(\bigcup_{i=1}^{n}(x_1^i, \cdots, x_{m_i}^i)\right)$$

of $S$. $T$ is finite and consists, say, of $k$ elements. Hence $Q=(T\backslash(x_1, \cdots, x_n))\cup(y_1, \cdots, y_{n-1})$ has $k-1$ elements. If $\sigma$ is the natural homomorphism of $G$ onto $G/pG$, this implies that $\sigma(T)$ which by the proof of (I) implies (II) forms part of a basis of $G/pG$ consisting of $k$ elements is dependent in $G/pG$ on the $k-1$ elements $\sigma(Q)$ which is impossible. This contradiction proves our assertion.

**Finally, we prove that** (VII) **is equivalent to** (V).

Assume that $G$ is infinite, and that it has a direct summand as specified in (V). Then it is easy to verify that the power of the automorphism group of $G$ is equal to $2^{|G|}$. Next assume (V) is not satisfied. If $|G|>|G/D|$ the conclusion is obvious. If not, we may assume $G$ is reduced, and by the equivalence of (IV) and (V), the power of $G$ would exceed that of any of its basic subgroups. Then an unpublished theorem of E. Walker, delivered to the author by oral communication, see [4, p. 867], asserts that the power of the automorphism group of $G$ is $|G|$. This also proves that (V) is equivalent to (VIII).

With this the proof of Theorem 1 is complete.

REMARK. From the proof of the equivalence of (I) and (IV) of Theorem 1, Lemma 3 can be stated as follows:

LEMMA 3a. *Let $G$ have a* (*M.s.g.*). *Then $x\in G$ can be extended to a* (*M.s.g.*) *of $G$ iff $h(x)=0$.*

Definition. Let $G$ be called a *starred* group if it satisfies any one of the eight equivalent conditions of Theorem 1. The following corollaries are obtained from Theorem 1 using well-known properties of abelian groups.

Corollary 1. *A primary divisible group $G$ is not a starred group.*

Corollary 2. *Let $B$ be basic in $G$, and $H \supset B$ be a subgroup of $G$. Then if $|H| = |B|$, $H$ is starred. Further, if $H$ is pure in $G$, then $H$ is starred only if $|H| = |B|$. In particular, if $G$ itself is starred then any subgroup of $G$ containing $B$ is starred.*

Corollary 3. *There exists a primary group without elements of infinite height which is not starred.*

Corollary 4. *Any countable reduced primary group is starred.*

Corollary 5. *Let $G$ be a reduced primary group and $B$ be basic in $G$. Assume $|B| = |B|^{\aleph_0}$. Then $G$ is starred.*

Corollary 6. *Let $G$ be reduced, and such that $n^{\aleph_0} < |G|$ for any cardinal $n$ less than $|G|$. Then $G$ is starred.*

Corollary 7. *Let $G$ be infinite. Then $G$ is starred iff the reduced part of $G$ is the direct sum of $|G|$ nontrivial subgroups of $G$.*

L. Fuchs asks the following question—see [1, Problem 18, p. 144]:

"Which are the cardinals $m$ such that there exist no $m$-indecomposable reduced $p$-groups (i.e., primary groups) of power $m$?"

We have

Theorem 2a. *There exists no infinite $m$-indecomposable reduced primary group of power $m$ if and only if either $m = \aleph_0$ or for every cardinal $n < m$ we have $n^{\aleph_0} < m$.*

Since by Corollary 7 for reduced groups of infinite cardinal $m$, $m$-decomposability is equivalent to the property of being starred, we may prove instead the following version of Theorem 2a:

Theorem 2. *There exists an unstarred reduced group $G$ of infinite cardinal $m$ if and only if there exists an infinite cardinal $n < m$ such that $n^{\aleph_0} \geqq m$.*

**Proof.** If $m = \aleph_0$, Corollary 4 implies that $G$ is starred. If $m > \aleph_0$ and for no infinite cardinal $n < m$ is $n^{\aleph_0} \geqq m$, then Corollary 6 implies that $G$ is starred.

Next assume $m$ is a cardinal for which there exists an infinite cardinal $n < m$ such that $n^{\aleph_0} \geqq m$. Then $m > \aleph_0$. Let $B_i$ be the direct sum of $n$ cyclic groups of order $p^i$. Let $\bar{B}$ be the closure of $B = \oplus_{i=1}^{\infty} B_i$. Then $|B| = n\aleph_0$, $\bar{B}$ is reduced (and even has no elements of infinite height) and $|\bar{B}| = n^{\aleph_0} \geqq m > n\aleph_0$. Also $\bar{B}/B$ is divisible and of cardinal $\geqq m$ since $|B| \leqq n\aleph_0$. Then there exists a pure subgroup $B^*/B$ of $\bar{B}/B$ (where $B^* \supset B$) of cardinal $m$. Then $|B^*| = m$, and $B$ is basic in $B^*$, which by IV of Theorem 1 means $B^*$ is not starred.

This concludes the proof of Theorem 2.

Problem 14 in L. Fuchs' book *Abelian groups* reads as follows: "Under what conditions can every element of a $p$-group be embedded in a direct summand of power $\leqq m$, $m$ an infinite cardinal?"

In this connection we have

THEOREM 3. *Let $G$ be an infinite starred group, and let $x \in G$ be contained in some pure subgroup of $G$ having no elements of infinite height. Then $x$ can be embedded in a direct summand $H$ of $G$ of power $m$ for any $m$ satisfying $m \leqq |G|$.*

**Proof.** It is known that such an $x$ can be embedded in a finite direct summand $K$ of $G$. Then let $G = K \oplus L$, and let $B$ be a basic subgroup of $L$. Then $|B| = |L|$, and by IV of Theorem 1 $L$ is starred. By V of Theorem 1, $L = L_1 \oplus L_2$ where $|L_1| = |L| = |G|$ and $L_1$ is a direct sum of cyclic groups. If $|G| \geqq m$, then clearly $L_1$ has a direct summand $L_3$ satisfying $|L_3| = m$. We may then set $H = K \oplus L_3$.

REMARK. If any $x \in G$ which is contained in some pure subgroup of $G$ having no elements of infinite height is also contained in some direct summand of $G$ having power $m$ for any $m \leqq |G|$, then it does not follow that $G$ is starred.

To see this we let $G$ be the direct sum of a countable number of groups each isomorphic to $\overline{B}$ of the proof of Theorem 2 with $n = 1$.

The following theorem is included for the sake of completeness:

THEOREM 4. *Let $G$ be a primary $p$-group. Then $G$ is a direct sum of cyclic groups of bounded order if and only if there is some basis of $G/pG$ for which any system of representatives by elements of $G$ generates $G$. If every system of representatives of some basis of $G/pG$ generates $G$, then every system of representatives of every basis of $G/pG$ generates $G$.*

**Proof.** Assume first that $G = \oplus_{i=1}^{n} B_i$ where each $B_i$ is a direct sum of cyclic groups $\{v_{a_i}\}_{a_i \in A_i}$ of order $p^i$, and $n$ is an integer. For any primary $p$-group $H$, let $S(H)$ denote the subgroup of $H$ consisting of those elements of $H$ whose orders are $p$ or 1. For any subgroup $K$ of $G$ let $K^i = p^i K$. Then $0 = G^n \subset G^{n-1} \subset \cdots \subset G^0 = G$, and $G^i = B_{i+1}^i \oplus \cdots \oplus B_n^i$. Also let $\sigma_i$ be the natural homomorphism of $G$ onto $G/G^i$. Then $(\sigma_1(v_{a_i})$, all $a_i \in A_i$ and $i = 1, \cdots, n)$ is a basis of $G/pG$, and any system of representatives of it is of the form $L = (v_{a_i} + p(g_{a_i})$, all $a_i \in A_i$ and $i = 1, \cdots, n$, and where $g_{a_i}$ is some element of $G)$. We will show that $\{L\} \supset S(G) = G^{n-1} = B_n^{n-1}$. Let $x \in B_n^{n-1}$; then $\exists v_{a_1 n}, \cdots, v_{a_m n}$ and integers $l_1, \cdots, l_m$ such that $x = l_1 v_{a_1 n} + \cdots + l_m v_{a_m n}$ where $p^{n-1}$ divides each $l_i$. Then also

$$x = l_1(v_{a_1 n} + p(g_{a_1 n})) + \cdots + l_m(v_{a_m n} + p(g_{a_m n}))$$

since $p^n y = 0$ for any $y \in G$. Thus $x \in \{L\}$. Thus $\{L\} \supset B_n^{n-1} = G^{n-1}$. Similarly $\{\sigma_{n-1}(L)\} \supset S(G/G^{n-1})$. Thus $\{L\} \supset G^{n-2}$. In the same way, having shown that $\{L\}\supset G^m$, it follows that $\{\sigma_m(L)\}\supset S(G/G^m)$, and hence that $\{L\}\supset G^{m-1}$. After a finite number of steps we arrive at $\{L\}\supset G^0=G$. Thus if $G$ is a bounded direct sum of cyclic groups, then there is a basis of $G/pG$ such that any system of representatives of this basis generates $G$.

Next we assume that there is a basis $(\bar{v}_a)_{a\in A}$ of $G/pG$ for which any system of representatives generates $G$, and we prove that if $(\bar{w}_b)_{b\in B}$ is any other basis of $G/pG$ then any system of representatives $(w_b)_{b\in B}$ of it generates $G$. If $\bar{v}_a=l_n^a\bar{w}_{b_1}+\cdots+l_n^a\bar{w}_{b_n}$, then $l_1^a w_{b_1}+\cdots+l_n^a w_{b_n}$ is certainly a representative of $\bar{v}_a$. Then the set $(l_1^a w_{b_1}+\cdots+l_n^a w_{b_n})_{a\in A}$ generates $G$. Hence $(w_b)_{b\in B}$ also generates $G$.

Now we prove that if there is some basis of $G/pG$ for which any system of representatives generates $G$, then $G$ is a bounded direct sum of cyclic groups. By what we have just proved, any system of representatives of any basis of $G/pG$ generates $G$. Let $B=\oplus_{a\in A}\{v_a\}$ be a basic subgroup of $G$. Then it is well known that if $\sigma$ is the natural homomorphism of $G$ onto $G/pG$, $(\sigma(v_a))_{a\in A}$ is a basis of $G/pG$. If we let $v_a$ represent $\sigma(v_a)$ we obtain that $G=B$, or that $G$ is a direct sum of cyclic groups. We must still show that $G$ is bounded. If $G$ were not bounded, there would exist $a_1, a_2, a_3, \cdots\in A$ such that order $v_{a_i}=p^{n_i}$, where $n_{i+1}>n_i$ for $i=1, 2, 3, \cdots$. Then $(v_a, a\in A, a\neq a_i$ for any $i)$ $\cup(v_{a_i}+p^{n_{i+1}-n_i}v_{a_{i+1}}, i=1, 2, 3, \cdots)$ certainly represents a basis of $G/pG$, and as is well known—this is also easy to see—does not generate $G$. The proof of Theorem 4 is now complete.

II. **On torsion groups.**

Notation. Let $P$ denote the set of primes.

Definition. If $T$ is an Abelian torsion group and $p\in P$, let $T_p$ be the $p$-component of $T$. A subgroup $B$ of $T$ is said to be a *basic subgroup* of $T$ if and only if $B_p$ is a basic subgroup of $T_p$ for every $p\in P$.

Theorem 5. *Let $T$ be an Abelian torsion group. Then the following statements are equivalent.*

(I) *$T$ has a minimal system of generators.*

(II) *$T$ has the same power as a basic subgroup of itself.*

(III) *$T$ has a direct summand which is a direct sum of cyclic groups and has the same power as $T$.*

**Proof.** First we prove that I implies II. So let $B=\oplus_{a\in A}\{v_a\}$ be a basic subgroup of $T$, and recall that all basic subgroups of a group are isomorphic. Let $S$ be a (m.s.g.) of $T$. With each $v_a$ associate a finite subset $S_a$ of $S$ such that $v_a\in\{S_a\}$. If we now assume that $|B|<|T|$, we obtain that $B\subset\{\cup_{a\in A}S_a\}<T$. Then if $\sigma$ is the natural homomorphism of $T$ onto $T/\{\cup_{a\in A}S_a\}$, $\sigma(S\backslash\cup_{a\in A}S_a)$ is a (m.s.g.) of $T/\{\cup_{a\in A}S_a\}$ which is divisible since it is a homomorphic image of $T/B$, and $T/B$ is divisible since each $T_p/B_p$ is. Thus it remains to show that no (non-zero) divisible torsion group $D$ has a (m.s.g.) $Q$. Let $x\in Q$; then $D>\{Q\backslash(x)\}$, and $\sigma(x)$ is a (m.s.g.) of $D/\{Q\backslash(x)\}$ where $\sigma$

is the natural homomorphism of $D$ onto $D/\{Q\backslash(x)\}$. But $D/\{Q\backslash(x)\}$ cannot be generated by one element since being a homomorphic image of $D$ it is itself divisible. Thus I implies II.

Trivially III implies II.

Next we assume that $|T| = |B|$ for some basic subgroup $B$ of $T$ and show that $T$ has a (m.s.g.), showing in the process that II implies III, i.e., we go from II to I via III.

*Case* 1. For each prime $p$, $B_p$ is finite. If $B_p \neq 0$, for only a finite number of primes $p_1, \cdots, p_n$, then $T = \oplus_{i=1}^{n} B_{p_i}$, and the result follows. In the other case, $T_p = B_p \oplus D_p$ where $D_p$ is divisible. Then necessarily $\aleph_0 = |\oplus_{p\in P} B_p| \geqq |\oplus_{p\in P} D_p|$. Let $\oplus_{p\in P} B_p = \oplus_{a\in A}\{v_a\}$, where the $v_a$'s have prime power order. Then since for each $p\in P$, $|B_p| < \aleph_0$, we may map the elements of prime power order of $\oplus_{p\in P} D_p$ in a one-one fashion on a subset $(v_a)_{a\in B\subset A}$ of $(v_a)_{a\in A}$ such that no element of $D_p$ is mapped on an element of $B_p$. For a justification of this see the proof of case 3. Then the set $(v_a + w_a)_{a\in A}$ where $w_a$ is the element mapped on $v_a$ if there is such a one, and $w_a$ is zero otherwise is easily seen to be a (m.s.g.) of $T$.

*Case* 2. For some $p_0 \in P$, there is an infinite $B_{p_0}$ with $|B_{p_0}| = |T|$. In this case $|B_{p_0}| = |T_{p_0}|$, and by V of Theorem 1, $T_{p_0} = T^1 \oplus T^2$ where $T^1$ is an infinite direct sum of cyclic groups $\oplus_{a\in A}\{v_a\}$, and $|T| = |T^1| \geqq |T^2|$. Then we may write $A$ as the union of two disjoint subsets $A_1$ and $A_2$ with $|A_1| = |A_2| = |A| = |T|$. By V of Theorem 1, $T^2 \oplus (\oplus_{a\in A_1}\{v_a\})$ has a (m.s.g.) $S$. Then we may map the elements of $\oplus_{p_0\neq p\in P} T_p$ in a one-one fashion on a subset of $(v_a)_{a\in A_2}$. Then the set $S \cup (v_a + w_a)_{a\in A_2}$ where $w_a$ is the element mapped on $v_a$ if there is such a one and $w_a$ is zero otherwise is easily seen to be a (m.s.g.) of $T$, and even a (M.s.g.) of $T$.

*Case* 3. Neither Case 1 nor Case 2 holds. Then there is no greatest $|B_p|$. For each $p\in P$, let $B_p = \oplus_{i=1}^{\infty} B_p^i$, where $B_p^i$ is a direct sum of cyclic groups of order $p^i$. Then one can verify that there is a subsequence $p_1, p_2, p_3, \cdots$ of the sequence of primes and integers $i_1, i_2, i_3, \cdots, i_j$ depending on $P_j$ such that $\sum_{j=1}^{\infty} |B_{p_j}^{i_j}| = |T|$, and $|B_{p_j}^{i_j}| < |B_{p_{j+1}}^{i_{j+1}}|$ for $j = 1, 2, 3, \cdots$. Since each $B_p^i$ is a direct summand of $T_p$, we have $T = T^1 \oplus \oplus_{j=1}^{\infty} B_{p_j}^{i_j}$. Let $\oplus_{j=1}^{\infty} B_{p_j}^{i_j} = \oplus_{a\in A}\{v_a\}$, where each $v_a$ belongs to some $B_p$. Divide $A$ into a sequence of disjoint subsets $A_1, A_2, A_3, \cdots$ such that $|A_k| = |A| = |T|$ for each integer $k$. Then for each integer $k$, $|(v_a)_{a\in A_k} \cap T_{p_k}| < |T|$. Thus if $q_1, q_2, q_3, \cdots$ is the sequence of primes, we may map the elements of $T_{q_l}^1$ in a one-one fashion onto a subset of $((v_a)_{a\in A_l} \backslash T_{q_l})$. Then the set $(v_a + w_a)_{a\in A}$ where $w_a$ is the element mapped on $v_a$ if there is such a one and $w_a$ is zero otherwise can be verified to be a (m.s.g.) of $T$.

This completes the proof of Theorem 5.

THEOREM 6. *Let T be an Abelian torsion group. Then the following statements are equivalent*:

(I) *T has a minimum system of generators.*

(II) *$T$ is finite or some basic subgroup of some primary component of $T$ has the same power as $T$.*

(III) *$T$ is finite or $T$ has a direct summand of the same power as $T$ which is a direct sum of cyclic groups whose orders are powers of the same prime.*

**Proof.** The theorem is easily verified for finite groups. Thus we may assume in the following that $T$ is infinite. First we prove that I implies II. So assume that for no $p\in P$ is $|B_p| = T$, and let $S$ be a (M.s.g.) of $T$. Let $Q$ be the set of primes such that $|T_q| < |T|$ for each $q\in Q$. For each $x\in T$, let $x=x_{p_1}+ \cdots +x_{p_n}$, where $x_{p_i}\in T_{p_i}$, the $p_i$'s being different primes, and let $P_x=(p_1, \cdots, p_n)$. Then we assume that $|(x\in S$ such that $p\in P_x)| \leqq |T_p|$ for each $p\in Q$. Now let $x\in S$ be written as $x=x_{p_1}+ \cdots +x_{p_n}$ as above. Then we may assume that $(P_1, \cdots, P_m)\subset P\backslash Q$, and $(P_{m+1}, \cdots, P_n)\subset Q$ for some integer $m$. Now since for $P_{m+j}$, $j=1, \cdots, n-m$, $|T_{p_{m+j}}| < |T|$, there exist $n-m$ different elements $y_{(m+j)}\in S$ with $y_{p_{m+j}}=0$, where the index $(m+j)$ in brackets does not mean that $y_{(m+j)}\in T_{m+j}$. Let $\bar{y}_{(m+j)}=y_{(m+j)}+x_{p_{m+j}}$. Moreover, since for $i=1, \cdots, m$, $|B_{p_i}| < |T|$, and $|T_{p_i}| = |T|$, by Theorem 1 $T_{p_i}$ has no (m.s.g.). But $T_{p_i}\subset \{(x_{p_i})_{x\in S}\}$. Thus there exist $m$ different elements $y_{(p_i)}\in (S\backslash(y_{(m+1)}, \cdots, y_{(n)}, x))$ such that for $i=1, \cdots, m$, $T_{p_i} \subset \{(x_{p_i})_{y_{(p_i)}\neq x\in S}\}$, where the index in brackets of $y_{(p_i)}$ does not mean that $y_{(p_i)}\in T_{p_i}$. Then if $y_{(p_i)}=y_{p_1}+ \cdots +y_{p_i}+ \cdots +y_{p_k}$, let $y_{(i)}=y_{p_1}+ \cdots +y_{p_{i-1}}+x_{p_i}+y_{p_{i+1}}+ \cdots +y_{p_k}$.

Then the set

$$[(y_{(1)}, \cdots, y_{(m)}, \bar{y}_{(m+1)}, \cdots, \bar{y}_{(n)}) \cup (S\backslash(x, y_{(p_1)}, \cdots, y_{(p_m)}, y_{(m+1)}, \cdots, y_{(n)}))]$$

is easily seen to generate $G$. But this set was obtained from $S$ by replacing $n+1$ elements of $S$ by $n$ elements of $G$. This contradicts the minimum property of $S$. Thus I implies II. The proof of II implies III follows easily from Theorem 1. The proof of III implies I is contained in Case 2 of the proof of Theorem 5—see the last sentence of the proof of that case.

The following theorem is partly included in results obtained by W. R. Scott, [5, p. 19–22], and is included here as an illustration of the applicability of Theorems 5 and 6.

THEOREM 7. *Let $T$ be an Abelian torsion group having a minimal system of generators and having one infinite primary component. Then $T$ has $2^{|T|}$ different subgroups all isomorphic to $T$; for every cardinal $n \leqq |T|$, $T$ has $2^n$ different isomorphic subgroups of cardinal $n$; $T$ has $2^{|T|}$ different automorphisms mapping some fixed basic subgroup of $T$ onto itself, and $T$ has $2^{|T|}$ different direct summands.*

**Proof.** By III of Theorem 5, $T=H\oplus K$ where $|H| = |T|$, and $H$ is a direct sum of cyclic groups.

To prove the first part, it is easy to establish, because of the particularly simple structure of $H$, that for any $n \leqq |T|$, $H$ has $2^n$ different isomorphic subgroups $(H_a)_{a\in A}$ of power $n$, all isomorphic to $H$ if $n = |T|$. If $n = |T|$, then $(H_a + K)_{a\in A}$ are the desired $2^{|T|}$ different subgroups isomorphic to $T$.

To prove the second part, let $B$ be a basic subgroup of $K$. Again because of the simple structure of $H$, it is easy to establish that $H$ has $2^{|T|}$ different automorphisms $(\sigma_a)_{a\in A}$. Let $\bar{\sigma}_a$ be the automorphism of $T$ which coincides with $\sigma_a$ on $H$ and maps $K$ identically on itself. Then $(\bar{\sigma}_a)_{a\in A}$ is a set of $2^{|T|}$ automorphisms of $T$ all of which map $H \oplus B$ which is a basic subgroup of $T$ onto itself. The proof of the last statement is obvious.

THEOREM 8. *Let $G$ be an infinite reduced primary group, and $B$ be a basic subgroup of $G$. Then the power of the automorphism group of $G$ is $2^{|B|}$.*

**Proof.** E. Walker proved this result in the case $|B| < |G|$; see [4, p. 867]. If $|B| = |G|$, then Theorem 7 applies.

THEOREM 9. *Let $T$ be an Abelian torsion group. Then $T$ can be embedded as a direct summand in an Abelian torsion group $H$ with a minimum system of generators (which is a fortiori a minimal system of generators) satisfying $|H| = |T|$.*

**Proof.** If $T$ is finite, set $H = T$. If $T$ is infinite, then let $K$ be a direct sum of $|T|$ finite cyclic groups whose orders are powers of the same prime. Set $H = T \oplus K$. Theorem 6 completes the proof.

THEOREM 10. *Let $T$ be an Abelian torsion group. Then $T$ can be embedded as a fully invariant subgroup in a group $H$ having a minimum system of generators (which is a fortiori a minimal system of generators) and having the same cardinal as $T$; more precisely, if $T$ is infinite, then $T$ can be embedded as a fully invariant subgroup in a group $H$ having a minimum system of generators such that $T = nH$ for any prescribed integer $n > 1$.*

**Proof.** If $T$ is finite set $H = T$. If $T$ is infinite we proceed as follows: There exists a free Abelian group $F = \oplus_{a\in A}\{v_a\}$ with $|F| = |T|$ and a subgroup $K$ of $F$ such that $F/K \cong T$. Consider another set $(w_a)_{a\in A}$ of free generators. Identify each $v_a$ with $nw_a$, and thus $F$ with $n\{(w_a)_{a\in A}\}$. Let $L = \{(w_a)_{a\in A}\}/K$. Then clearly $nL = T$, and $|L| = |T|$. Now let $p$ be a prime dividing $n$, and let $Z$ be a direct sum of $|T|$ cyclic groups of order $p$. Set $H = Z \oplus L$. Then $|H| = |T|$, and by Theorem 6 $H$ has a (M.s.g.). Then also $nH = nZ \oplus nL = nL = T$, which implies the full invariance of $T$ in $H$.

COROLLARY 8. *Let $G$ be a primary group. Then $G$ can be embedded as a direct summand in a starred (primary) group $H$ satisfying $|H| = |T|$.*

COROLLARY 9. *Let $G$ be a $p$-group. Then $G$ can be embedded as a fully invariant subgroup in a starred group $H$ of the same cardinal, more precisely; if $G$ is* *infinite then* $G$ *can be embedded as a fully invariant subgroup in a starred group* $H$ *of the same cardinal such that* $G = pH$.

REMARK. If in Corollary 8, $H = K \oplus G$ where $K$ is a direct sum of cyclic groups of order $p$, then this embedding coincides with that of Corollary 9 in the sense that $pH = G$ if and only if $G$ is divisible, for $G = pH = pK \oplus pG = pG$, implies that $G$ is divisible.

REMARK. W. R. Scott suggested generalizing the above results to modules over countable rings. It turns out that most of the above theorems and in particular Theorem 1 hold for torsion modules over a principal ideal ring or over a complete discrete valuation ring, after the customary changes in terminology—such as substituting finitely generated for finite, etc.—are made; see [1, p. 29].

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